Engineering Notes

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Induced Drag and the Ideal Wake of a Lifting Wing†

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†The editors of the Journal recognize that this article is controversial. However, in the spirit of providing a forum for authors to express new ideas or to give alternate interpretations of old ideas, we are publishing it.

Nomenclature

a	= rearward component of wake velocity relative to fluid at infinity
a_0	= rearward component of fluid velocity relative to fluid at infinity at a plane upstream of the wing
$dE_{(front)}/dt$	= rate of accumulation of energy in the part of the wake forward of a plane fixed relative to fluid at infinity
dI/dt	= rate at which impulse accumulates in the wake
D_i	= induced drag
$D_{i,m}$	= force on control volume due to momentum change
$D_{i,m\infty,\infty}$	= induced drag due to momentum change
$D_{i,p}$	= pressure force on control volume
$D_{i,p\infty,\infty}$	= induced drag due to pressure
$Flux_{E(forward)}$	= net flow of energy from the rear of the wake to the front past a plane fixed relative to fluid at infinity
$Flux_{E(hydraulic,\ forward)}$	= forward flow of energy in the wake as mechanical work past a plane fixed relative to fluid at infinity
$Flux_{KE(aft)}$	= aft flow of kinetic energy in the wake past a plane fixed relative to fluid at infinity
p	= pressure in the wake relative to that at infinity
p_0	= pressure relative to that at infinity at a plane upstream of the wing
r	= a position vector from any fixed center
s	= distance along the perimeter of a transverse cross section of the wing or other body
v	= lateral component of wake velocity
v_0	= lateral component of velocity at a plane upstream of the wing
V	= speed of the wing relative to fluid at infinity
w	= downward component of wake velocity
w_0	= downward component of fluid velocity at a plane upstream of the wing
x	= distance aft of an arbitrary plane that moves with the wing
β	= angle at which the wake is inclined downward
ϕ	= slope of the outer surface of a body relative to the x axis, the sign chosen so the outward normal slopes upstream
ρ	= density of the fluid
ω	= vorticity

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Introduction

A FAMILIAR approximation to the vortex wake of a wing is a plane vortex sheet connecting the bound vorticity at the wing with the starting vortex. The sheet slopes downward toward the rear. The growth of wake impulse with time, directed downward and slightly forward, accounts quite well for the lift of the wing, but, as found by Sears, the forward component is twice the induced drag.[1] Applying Bernoulli's theorem to the assumed wake, Sears finds that the integral of the pressure over the cross section of the wake is positive and equal to one-half the momentum drag due to the forward component of wake impulse, so the net induced drag agrees with the value derived in other ways. Sears was surprised to find that the integral of pressure was positive. He said, "Thus one's usual concept—at least the author's—wherein the drag is balanced by reduced pressures downstream, seems to be incorrect."

As the wake force exists in the Trefftz plane, which is anywhere and everywhere more than a few spans behind the wing, Sears would have a compressive stress throughout the length of the wake, making it a long column under compression. Why does it not buckle?

And what contains its positive pressure? The pressure in a vortex is lower than the pressure outside. For it to be higher, the circumferential velocity in the vortex would have to be imaginary so as to reverse the direction of the centrifugal field.

The sloping vortex sheet is very different from the inviscid idealization of the real wake. In the sheet model, all motion is at right angles to the long axis of the sheet. Using such arbitrarily specified velocities in Bernoulli's theorem will not yield true pressures. The idealized wake results from the rolling up of the sheet and consists of two regions of roughly helical vorticity. Within each tip vortex, fluid flows aft.[2] Figure 1 shows the vortex wake of a lifting wing. In the sketch, the vortex sheet is not cut off at the rear in a plane, but at the locus of fluid particles that had once all been in the same transverse plane upstream of the wing. In vortex cores, this locus is displaced to the rear by rearward axial flow. One vortex line is traced on the sheet on each side. The innermost turns of the sheet in each vortex are not shown. Helical vorticity makes each vortex analogous to a coaxially nested array of solenoids wound at large pitch with many conductors in parallel. (An infinitely long, hollow, current-carrying solenoid has a circumferential magnetic field outside it and an axial magnetic field within.) In the following, it is shown that the aft flow drastically alters the picture given by Sears.

The treatment makes no assumptions about the flow pattern. It depends only on the laws of physics. These laws enforce the pattern shown in Fig. 1.

Idealized Theory

In the coordinate system of the wing, the flow is steady, so the fluid obeys Bernoulli's theorem everywhere. Because there is no viscosity, there is no separation and no viscous wake. Upstream of the wing, erect a plane perpendicular to the velocity of fluid at infinity and thus also perpendicular to the path of the wing. On this plane, the pressure relative to that in fluid at infinity is called p_0 and given by

$$p_0 = (\rho/2)(V^2 - V^2 - 2Va_0 - a_0^2 - v_0^2 - w_0^2)$$
$$= (\rho/2)(-2Va_0 - a_0^2 - v_0^2 - w_0^2) \qquad (1)$$

The longitudinal velocity of the fluid is $V + a_0$, the speed V of the wing plus the rearward velocity a_0 (for "added") of the fluid relative to fluid at infinity; v_0 is the lateral, w_0 the vertical velocity, and ρ the density of the fluid.

Likewise, on a plane downstream of the wing parallel to the upstream plane, the pressure p is given by

$$p = (\rho/2)(V^2 - V^2 - 2Va - a^2 - v^2 - w^2)$$
$$= (\rho/2)(-2Va - a^2 - v^2 - w^2) \qquad (2)$$

Join the planes by a cylindrical surface, with the flight path as axis, to complete a control volume around the wing. Because the cylindrical surface is parallel to the flight path, pressure upon it causes no force in the direction of motion. The pressure force $D_{i,p}$ on the control volume is, therefore, the difference between the integrals of pressure over the up- and downstream planes, identified by the elemental areas dA_0 and dA, respectively. Letting the radius of the control volume increase without limit,

$$D_{i,p} = (\rho/2)\int(-2Va_0 - a_0^2 - v_0^2 - w_0^2)\,dA_0$$
$$-(\rho/2)\int(-2Va - a^2 - v^2 - w^2)\,dA \qquad (3)$$

the integrals being over the infinite planes.

From continuity,

$$\int a_0\,dA_0 = \int a\,dA \qquad (4)$$

because the right side of the equation minus the left equals the influx through the cylindrical surface, which, from the following train of argument, is zero in the limit as the radius goes to infinity. Along any radius drawn from the wing, the velocity induced by the semi-infinite vortex pair falls as the square of the distance, if this is large compared to the spacing between the vortices and the radius chosen does not point back along the pair. This is true even though the vortices in the pair are helical. The far field of a semi-infinite solenoid of negligible pitch is, in effect, produced by a source at the end of the solenoid. The velocity induced by the bound vortex also falls as the square of distance if the distance is large. Because, along any radius, the transverse velocity goes as $(\text{distance})^{-2}$, whereas the area of the cylindrical surface rises as $(\text{distance})^1$, the influx through the cylindrical surface falls with increasing radius at least as fast as $(\text{distance})^{-1}$, and Eq. (4) holds in the limit as the radius approaches ∞.

Combining Eqs. (3) and (4) gives

$$D_{i,p} = (\rho/2)\int(-a_0^2 - v_0^2 - w_0^2)\,dA_0$$
$$+(\rho/2)\int(a^2 + v^2 + w^2)\,dA \qquad (5)$$

Because a_0^2, v_0^2, and w_0^2 fall off as $(\text{distance})^{-4}$, the first integral in Eq. (5) may be made arbitrarily small by moving the A_0 plane far enough upstream. The second integral depends on the location of the downstream plane of integration, but far enough downstream so the vortex sheet has rolled up into a vortex pair, i.e., in the Trefftz region, there is little change with distance, hence,

$$D_{i,p\infty,\infty} = (\rho/2)\int(a^2 + v^2 + w^2)\,dA \qquad (6)$$

is the part of the induced drag that is due to pressure. The two infinites in the subscript mean that Eq. (6) is the limiting case as the up- and downstream planes are moved farther and farther from the wing. [Multiplying the integrand in Eq. (6) by $(V + a)\,dA$, gives the rate at which the negative wake pressure, acting over the elemental area dA, does work on the control volume and the identical rate at which kinetic energy leaves the control volume through the same elemental area dA.]

To calculate $D_{i,m}$, the rate at which the fluid adds downstream momentum to the control volume, note first that the flux of momentum across the cylindrical surface is always smaller than $(V + a)^2$ times the transverse slope of the streamlines, which is the quotient of transverse by longitudinal velocity. Since transverse velocity goes as $(\text{radius})^{-2}$ at large radius, the flux of momentum through the cylindrical surface goes as $(\text{radius})^{-1}$ and $\rightarrow 0$ as the radius $\rightarrow \infty$. So in the limit as the radius $\rightarrow \infty$, the momentum flow into the control volume is the difference between the momentum fluxes across the up- and downstream planes. The momentum that enters but does not leave the control volume is the momentum contribution to the force $D_{i,m}$ on the control volume. Therefore,

$$D_{i,m} = \rho\int(V^2 + 2Va_0 + a_0^2)\,dA_0 - \rho\int(V^2 + 2Va + a^2)\,dA$$
$$= \rho\int a_0^2 dA_0 - \rho\int a^2\,dA \qquad (7)$$

the last equality a consequence of applying Eq. (4). As before, the integral of a_0^2 vanishes in the limit if the upstream plane is moved far from the wing and the downstream integral is almost independent of distance, once the distance is large. So,

$$D_{i,m\infty,\infty} = -\rho\int a^2\,dA \qquad (8)$$

is the part of the induced drag due to longitudinal momentum, in coordinates fixed relative to fluid at infinity, that is removed from the fluid. Strictly, the measurable quantity that appears in the fluid is the Kelvin impulse. The Kelvin impulse of a flow pattern is the density of the fluid times half the first moment of its vorticity or $(\rho/2)\int r \times \omega d$ (volume), the integral taken over all space, where r is the position vector of the elemental volume relative to an arbitrary center and ω the vorticity at that elemental volume. Whereas the momentum of a flow pattern in incompressible fluid of infinite extent is not

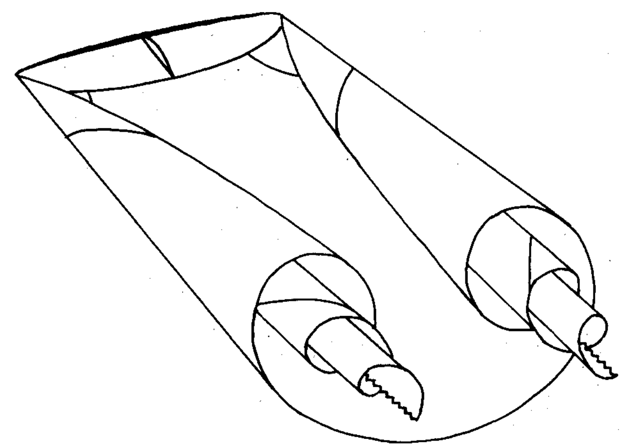

Fig. 1 The vortex wake of a lifting wing.

uniquely defined, the impulse needed to create the flow pattern is always given by this expression.[3]

Because the integral of a^2 is always positive, the momentum drag is always negative. This is true for a wing or any other object that disturbs the air, even in the presence of viscosity. In inviscid flow, the positive suction drag is $\int(a^2 + v^2 + w^2)\mathrm{d}A$ and is larger than the negative of momentum drag in proportion as $\int(a^2 + v^2 + w^2)\,\mathrm{d}A$ is larger than $2\int a^2\mathrm{d}A$.

The total induced drag could be found by adding $D_{i,p\infty,\infty}$ from Eq. (6) to $D_{i,m\infty,\infty}$ from Eq. (8). However, Lofquist[4] pointed out to me that the total drag must not depend on the location of bounding planes of the control volume, so long as they are clear of the wing. Adding Eqs. (5) and (7) gives the total induced drag for general positions of both planes,

$$D_i = (\rho/2)\int(a_0^2 - v_0^2 - w_0^2)\,\mathrm{d}A_0 + (\rho/2)\int(-a^2 + v^2 + w^2)\,\mathrm{d}A \tag{9}$$

The independence of D_i on the location of the planes requires that both integrals be independent of the position of the planes of integration so long as they miss the wing. Since all terms of the first integrand approach zero as the upstream plane is moved away from the wing, this integral is zero for any position of the plane; consequently,

$$D_i = (\rho/2)\int(-a^2 + v^2 + w^2)\,\mathrm{d}A = -\int(p + \rho a^2)\,\mathrm{d}A \tag{10}$$

[which is the sum of Eqs. (6) and (8)] for any positions of both planes that do not intersect the wing. [Eq. (10) is what Landau and Lifshitz[5] would have found their Eq. (47.3) to be, had they not neglected the $-a^2$ term. Subject to this approximation, their result is correct, although Sears is right that their derivation is faulty.] This independence of the position of the planes of integration is true of the total induced drag, not of its pressure and momentum constituents separately.[4]

Equation (9) is true even if the planes intersect the wing, the part of any plane lying within the wing, or other body being excluded from the integral. It gives the induced drag on the whatever bodies or parts of bodies lie between the planes. In the limit that the planes become very close together, $\mathrm{d}D_i/\mathrm{d}x$, the induced drag per unit length x of body or bodies, is given by the x derivative of $(\rho/2)\int(-a^2 + v^2 + w^2)\,\mathrm{d}A$, hence

$$\frac{\mathrm{d}D_i}{\mathrm{d}x} = \sum\nolimits_{\text{bodies}}\oint p\tan\phi\,\mathrm{d}s = \rho\int\left[-a\left(\frac{\partial a}{\partial x}\right) + v\left(\frac{\partial v}{\partial x}\right) + w\left(\frac{\partial w}{\partial x}\right)\right]\mathrm{d}A = -\int\left[\frac{\partial p}{\partial x} + 2\rho a\frac{\partial a}{\partial x}\right]\mathrm{d}A \tag{11}$$

where s is distance along the perimeter of a transverse cross section of a body, and ϕ is the slope of the surface of the body relative to the x axis, ϕ being positive when the outward normal slopes forward. The body or bodies may extend indefinitely in the plus or the minus x direction; thus, Eqs. (9) and (11) apply to a wind-tunnel model and its sting.

The presence of $-a^2$ in Eq. (10) looks wrong. Induced drag results from adding energy to the fluid, yet here is a term in velocity squared that reduces the drag. The explanation lies in a forward flow of energy in the wake. To see this, first calculate the rate at which energy, in the coordinate system fixed relative to fluid at infinity, accumulates in the part of the wake forward of a vertical plane in the Trefftz region behind the wing. The plane is fixed relative to fluid at infinity and oriented at right angles to the wing's motion. The rate of energy accumulation in advance of the plane is the speed V of the wing away from the plane times the energy per unit length of wake. So,

$$\mathrm{d}E_{(\text{front})}/\mathrm{d}t = (\rho/2)\int(a^2 + v^2 + w^2)V\,\mathrm{d}A \tag{12}$$

Subtract from this the drag power

$$D_iV = (\rho/2)\int(-a^2 + v^2 + w^2)V\,\mathrm{d}A \tag{13}$$

and the difference is

$$\mathrm{Flux}_{E(\text{forward})} = \rho V\int a^2\,\mathrm{d}A \tag{14}$$

the net rate of flow of energy from the rear of the wake to the front. It is always positive. Comparing Eq. (14) with Eq. (8) shows that $\mathrm{Flux}_{E(\text{forward})}$ equals the negative of the momentum drag power. The negative momentum drag, which is momentum thrust, is paid for by energy recovered from the wake.

$\mathrm{Flux}_{E(\text{forward})}$ is the net flux, the difference between a forward flux,

$$\mathrm{Flux}_{E(\text{hydraulic, forward})} = -\int pa\,\mathrm{d}A = (\rho/2)\int(2Va + a^2 + v^2 + w^2)a\,\mathrm{d}A \tag{15}$$

the rate at which energy flows forward as mechanical work done by the (mostly negative) pressure in the wake times the velocity of the fluid, and

$$\mathrm{Flux}_{KE(\text{aft})} = (\rho/2)\int(a^2 + v^2 + w^2)a\,\mathrm{d}A \tag{16}$$

the rate at which kinetic energy in the wake is carried aft across the plane by the wake fluid's aft velocity a. The forward flux is larger than the rearward flux.

At first sight, this appears paradoxical. How can the rear part of the wake provide energy? The answer is that it contains energy, some of which can be recovered. In free fluid, vortex lines can exist only as loops. The rear ends of the trailing vortices are connected to each other by the starting vortex. At the starting vortex, fluid flowing aft in the core of one trailing vortex meets fluid flowing aft in the others and the aft region of the vortex loop swells to accommodate it. The fluid in the swelling vortex is decelerated as it climbs the pressure gradient, loses kinetic energy, and approaches zero velocity as the vortex grows toward infinite diameter.

What, in this idealized, inviscid world, does the drag force react against? At the Trefftz plane, induced drag results from negative pressure in the wake. The wake is in a state of tension. The rear end of the wake must be anchored. The only way to anchor something in fluid, imperfectly at that, is to have it create impulse. The rear end of the wake is a hairpin loop, which is convected downward and then forward. (This particular flight started with a catapult launch at cruising altitude.) The impulse of this forward-facing downward loop accounts for induced drag—and a bit more to balance the rearward impulse of the negative momentum drag. The impulse in the loop grows linearly with time if the wing is in steady flight.

The velocity field of this loop extends throughout all space. In the region around the wing, it causes the integrals of a_0 and a over the upstream and Trefftz planes to be nonzero, but its contributions to velocity and pressure at the two planes are almost identical. The pressure is minute, but distributed over an enormous area to give a nonzero force. The entire atmosphere is accelerated, so the pressure is positive everywhere forward of the starting vortex and negative behind it.

If a 747 airliner leaving New York were pulled from London by a weightless rope and there were no skin friction (and the Earth were flat), the tension in the rope would almost equal the tension in the wake, the difference being accounted for by the negative momentum drag. The tension in the wake would add London-aimed impulse to its New York end. Because this flight started with a takeoff, the New York end of the wake would not be high in the sky and convecting downward, but near the ground and split apart, each vortex convecting outward in the velocity field of its image beneath the flat landscape of Long Island. Instead of one hairpin loop, there would be two, each formed where a vortex joins its image, each curling forward and each losing energy as it fattens on the arriving axial flow.

Why does the tipped-sheet vortex wake correctly predict the lift, yet misrepresent the cause of drag? The tipped sheet is a superposition of rectangular vortex loops of various widths. In the real wake, the long sides of these loops are replaced by helices. Figure 2 shows top and rear views of a single vortex line in the horseshoe formed by the bound vortex and the two trailing vortices. The arrows show the hand of the vorticity. Seen from above or below, a length of real wake has about the same first moment and vertical component of impulse as the rectangle and therefore about the same lift. Seen from the front or rear, the moment of the helical trailing vortices opposes the moment of the rectangle and is larger, when all vortex filaments are considered. The momentum drag is therefore negative, whereas for the rectangular vortex loop it would be positive.

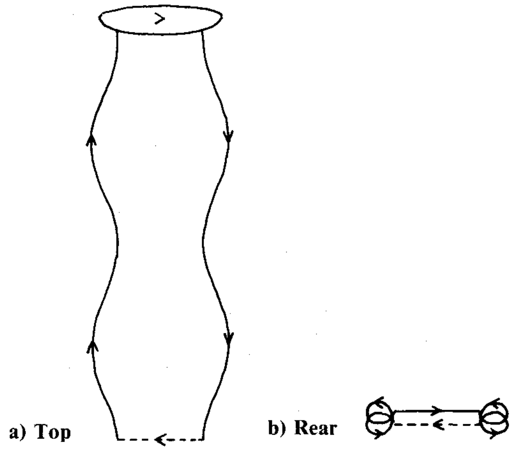

Fig. 2 Top and rear views of a single-vortex filament in the wake.

Practical Consequences

How big is the negative momentum drag $D_{i,m\infty,\infty} = -\rho\int a^2 \mathrm{d}A$ relative to the pressure drag $D_{i,p\infty,\infty} = (\rho/2)\int(a^2 + v^2 + w^2)\mathrm{dA}$? For a rough estimate, remember that the tip vortices are open at the front and suck in freestream fluid. The pressure drop due to longitudinal motion must be of the same order as that due to transverse motions. Thus, $V^2 - (V + a)^2 = -2Va - a^2$ must, in some average sense, be near $-v^2 - w^2$, which means that when v and w approach V in magnitude, a approaches v and w (Ref. 2 reaches a similar conclusion) and the negative momentum drag ceases to be negligible relative to the suction drag. This condition is met at high span loading. In fact, in the vortex above the leading edge of a delta wing at high angles of attack, v and w can exceed V in magnitude and a can exceed all three. Rearward flows several times the speed of the aircraft have been measured in these vortices in model experiments.[6,7]

The rearward flow in the trailing vortices affects lift as well as drag. Because it increases the speed with which shed vorticity is carried aft, it raises the rate $\mathrm{d}I/\mathrm{d}t$ at which impulse is shed into the wake. This rate is given by

$$\mathrm{d}I/\mathrm{d}t = \rho\int r \times \omega(V + a)\mathrm{d}A \qquad (17)$$

and is the vector negative of the total momentum force on the wing. Here, r is a vector from any fixed center and ω the vorticity. In addition, the axial flow contributes some of the downward impulse given by Eq. (17) by virtue of its own momentum, because of the downward inclination of the wake. The wake inclination β, however, also reduces lift in a way not accounted for in Eq. (17), because it directs wake suction downward. The reduction is $D_{i,p\infty,\infty} \tan\beta$.

In real life, skin friction accelerates the boundary layer forward and creates a sheet of forward-moving fluid in wake that gets wound into the tip vortices. At low span loading, the forward impulse of this fluid can be larger than the rearward impulse due to roll-up of the vortices. The latter slightly more than cancels the forward component of the impulse that results from the slope of the trailing vortices, which, as Sears notes,[1] is about twice the induced drag. Therefore, the forward impulse due to skin drag will exceed the rearward impulse due to roll-up whenever the skin drag of the wing is more than twice the induced drag. Since skin drag goes about as (speed)^2 and induced drag as (speed)$^{-2}$, the speed at which the skin drag is twice the induced drag is 2¼ times the speed that makes them equal, which is also the speed at which their sum is least. So, at any speed more than 2¼ times the speed of minimum wing drag, the forward impulse in the air due to skin drag will be greater than the rearward impulse due to wake roll-up.

The forward motion due to skin drag associated with a particular bit of vortex sheet in a tip vortex occurs at the sheet. The aft motion that the bit of sheet induces occurs at all points nearer the axis of the vortex. So, in the absence of some other phenomenon, there should be rearward motion near the axis even at low span loading when the overall wake motion is forward (the propulsive wake is here ignored). As the lift coefficient rises, the rearward flow should become more prominent, as, indeed, it did in the only experiment I know of that found rearward flow in the tip vortex of a conventional wing.[8]

Obviously, in a real fluid with viscosity and turbulence, rearward flow is not pumped by a lossless recovery of energy from the starting vortex end of the wake. Instead, it must be pumped by swirl energy much closer to the wing through some process involving dissipation. To the rear of where this process occurs, axial flow in the vortex cores will be forward. At high angles of attack, vortex breakdown can occur well within the pressure field of a delta wing, even ahead of the trailing edge, and can modify the forces on the wing.[9]

Summary

The wake of a lifting wing in inviscid flow is a pair of downward-sloping vortices with distributed vorticity in which the vortex lines are approximately helical. Within the vortices, fluid flows aft relative to fluid at infinity, and its Kelvin impulse more than nullifies the forward Kelvin impulse caused by the slope of the vortex pair. The momentum drag is therefore negative. Suction in the vortex cores more than counterbalances the negative momentum drag, and the difference between the two is the induced drag. The negative momentum drag is much smaller in magnitude than the drag due to suction except at high span loading. The induced drag is the integral over any transverse plane behind the wing of the difference between the energies of lateral and longitudinal motion per unit length. The drag contributed by the portion of the wing, or of any distribution of bodies, between any two transverse planes is equal to the growth of the above integral between the forward and after of these planes.

Acknowledgments

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Structural Optimization for Aeroelastic Control Effectiveness

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Introduction

THE issue of aeroelastic effectiveness of wing trailing-edge control surfaces may present a major design goal for the aircraft designer. A fully stressed and buckling-designed wing is sometimes deficient in meeting the required aircraft performance because of insufficient control effectiveness at high speed. The problem is how to resize the wing structure such that the control effectiveness requirements will be satisfied with minimum increase in weight.

The FASTOP computer program[1,2] applied a simple and efficient method for optimization of metallic structure with strength and flutter constraints. The method was later extended to deal with composite structures subjected to strength and deflection constraints and integrated into the ASOP-3 computer program.[3] Further development of the basic optimization procedure of FASTOP to address constraints on aeroelastic effectiveness and static divergence is given by Lerner and Markowitz[4] and was used in the design of the X-29 forward-swept-wing demonstrator aircraft[5] and on the Lavi wing and vertical tail.[6] The most time-consuming portion of the Ref. 4 procedure is the calculations of the aeroelastic effectiveness parameters after each optimization step. This calculation is performed using a discrete-coordinate approach with the structural model clamped to the ground to allow inversion of the stiffness matrix. As a result, inertia relief effects are not taken into account during the optimization process.

Sheena and Karpel[7] used the modal approach for performing static aeroelastic analysis. The present work follows the formulation of Ref. 7 and extends it to include effectiveness derivative calculations along the optimization path. This application, which uses free-free aircraft vibration modes, allows larger optimization steps between two finite-element model updates and makes the calculations so efficient that a major optimization cycle may be completed in a short online computer session. Another advantage of the present method is that inertia relief effects are implicitly taken into account in calculating the effectiveness parameters and their derivatives.

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Aeroelastic Effectiveness

The method of static aeroelastic analysis using aircraft vibration modes is described in Ref. 7. The basic assumption in the modal approach is that the deflections of the structure can be represented as a linear combination of a limited set of vibration mode shapes. When a set of free-free vibration modes is used as generalized coordinates, the aircraft matrix equation of motion can be partitioned into rigid and elastic parts:

$$\begin{bmatrix} M_{rr} & 0 \\ 0 & M_{ee} \end{bmatrix} \begin{Bmatrix} \ddot{\xi}_r \\ \ddot{\xi}_e \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{Bmatrix} \dot{\xi}_r \\ \dot{\xi}_e \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{ee} \end{bmatrix} \begin{Bmatrix} \xi_r \\ \xi_e \end{Bmatrix} = q \begin{bmatrix} R_{rr} & R_{re} \\ R_{er} & R_{ee} \end{bmatrix} \begin{Bmatrix} \xi_r \\ \xi_e \end{Bmatrix} + \begin{Bmatrix} F_r \\ F_e \end{Bmatrix} \tag{1}$$

where $[M]$, $[C]$, and $[K]$ are the diagonal generalized mass, damping, and stiffness matrices, $[R]$ is the generalized aerodynamic force matrix, $\{\xi\}$ is the generalized coordinate displacement vector, and $\{F\}$ is a vector of the generalized external forces which cannot be related to $\{\xi\}$ through a constant coefficient matrix. Subscripts r and e related to rigid-body and elastic modes, respectively, and q is the dynamic pressure. The elements of $[R_{rr}]$ are the rigid aerodynamic coefficients associated with rigid-body displacements. The generalized aerodynamic matrix is calculated by

$$[R] = [\Psi]^T[AFC][\Psi'] \tag{2}$$

where $[AFC]$ is the aerodynamic force coefficient matrix obtained by a linear panel aerodynamic theory, $[\Psi]$ is the mode shape matrix where displacements are defined at the panel centroids, and $[\Psi']$ are the chordwise derivatives of $[\Psi]$, where slopes are defined at the panel control points. $[\Psi]$ and $[\Psi']$ are obtained from the modal displacements at the structural points by a surface spline routine. When a quasisteady motion is assumed,

$$\{\ddot{\xi}_e\} = \{\dot{\xi}_e\} = \{0\} \tag{3}$$

and Eq. (1) yields

$$[M_{rr}]\{\ddot{\xi}_r\} = q[\bar{R}_{rr}]\{\xi_r\} + \{\bar{F}_r\} \tag{4}$$

where

$$[\bar{R}_{rr}] = [R_{rr}] + q[R_{re}]([K_{ee}] - q[R_{ee}])^{-1}[R_{er}] \tag{5}$$

$$\{\bar{F}_r\} = \{F_r\} + q[R_{re}]([K_{ee}] - q[R_{ee}])^{-1}\{F_e\} \tag{6}$$

The elements of $[\bar{R}_{rr}]$ of Eq. (5) are the "flexibilized" aerodynamic coefficients associated with the aircraft rigid-body displacements. The rigid-body coordinates can be extended to include rigid control surface rotations. In this way, Eq. (5) can be used to flexibilize aerodynamic coefficients such as rolling moment due to aileron deflection (C_{L_δ}) or hinge moment due to aileron deflection (C_{H_δ}). Equation (6) can be used to flexibilize aerodynamic coefficients due to rigid-body velocities. This is done by defining

$$\begin{bmatrix} F_r \\ F_e \end{bmatrix} = q\begin{bmatrix} \Psi_r^T \\ \Psi_e^T \end{bmatrix}[AFC]\{\alpha_i\} \tag{7}$$

where $\{\alpha_i\}$ is the vector of induced-panel angles-of-attack due to the rigid-body velocities.

The aeroelastic effectiveness of an aerodynamic coefficient is defined as

$$\eta_{ij} \equiv \frac{\bar{R}_{rr_{ij}}}{R_{rr_{ij}}} = 1 + \frac{q}{R_{rr_{ij}}}\{R_{re_i}\}^T([K_{ee}] - q[R_{ee}])^{-1}\{R_{er_j}\} \tag{8}$$

where $\{R_{re_i}\}^T$ and $\{R_{er_j}\}$ are the ith row and the jth column of $[R_{re}]$ and $[R_{er}]$, respectively.

Effectiveness Derivatives

It is assumed that the weight of a structural element w_k is